# Lecture 24: Noise Operator

# Noise Operator

- Let $N_\varepsilon$ be a the noise distribution over $\{0,1\}^n$ such that the probability $\mathbb{P}\left[N_\varepsilon = x\right] = \varepsilon^{|x|}(1-\varepsilon)^{n-|x|}$, where $|x|$ represents the number of 1s in $x$
- Intuitively, consider the noise operator starting with $0^n$ and flipping each input bit with probability $\varepsilon$; otherwise keeping it intact with probability $(1-\varepsilon)$

- We are interested in computing the Fourier coefficients of the function $N_\varepsilon$. We shall show the following result

**Lemma**

$$\widehat{N_\varepsilon}(S) = \frac{(1-2\varepsilon)^{|S|}}{N}$$

- We shall prove it in two different ways

  **Proof Outline.**

# Fourier Coefficients of the Noise Operator: First Technique II

- Let us start the calculation of the Fourier coefficient

$$\widehat{N_\varepsilon}(S) = \frac{1}{N} \sum_{x\in\{0,1\}^n} \varepsilon^{|x|}(1-\varepsilon)^{n-|x|}(-1)^{S\cdot x}$$

$$= \frac{(1-\varepsilon)^n}{N} \sum_{w=0}^{n} \sum_{\substack{x\in\{0,1\}^n \\ S\cdot x = w}} \left(\frac{\varepsilon}{1-\varepsilon}\right)^{|x|} (-1)^w$$

$$= \frac{(1-\varepsilon)^n}{N} \sum_{w=0}^{n} \sum_{k\geqslant 0} \sum_{\substack{x\in\{0,1\}^n \\ S\cdot x = w \\ |x| = w+k}} \left(\frac{\varepsilon}{1-\varepsilon}\right)^{|x|} (-1)^w$$

- Note that there are exactly $\binom{|S|}{w}\binom{n-|S|}{k}$ bit-strings $x \in \{0,1\}^n$ such that $S \cdot x = w$ and $|x| = w + k$

# Fourier Coefficients of the Noise Operator: First Technique III

- So, we simplify the above expression as

$$
\begin{aligned}
\widehat{N_\varepsilon}(S) &= \frac{(1-\varepsilon)^n}{N} \sum_{w=0}^{n} \sum_{k\geqslant 0} \binom{|S|}{w} \binom{n-|S|}{k} \left(\frac{\varepsilon}{1-\varepsilon}\right)^{w+k} (-1)^w \\
&= \frac{(1-\varepsilon)^n}{N} \sum_{w=0}^{n} \binom{|S|}{w} \left(\frac{-\varepsilon}{1-\varepsilon}\right)^{w} \sum_{k\geqslant 0} \binom{n-|S|}{k} \left(\frac{\varepsilon}{1-\varepsilon}\right)^{k} \\
&= \frac{(1-\varepsilon)^n}{N} \sum_{w=0}^{n} \binom{|S|}{w} \left(\frac{-\varepsilon}{1-\varepsilon}\right)^{w} \left(1+\frac{\varepsilon}{1-\varepsilon}\right)^{n-|S|} \\
&= \frac{(1-\varepsilon)^n}{N} \left(\frac{1}{1-\varepsilon}\right)^{n-|S|} \sum_{w=0}^{n} \binom{|S|}{w} \left(\frac{-\varepsilon}{1-\varepsilon}\right)^{w} \\
&= \frac{(1-\varepsilon)^n}{N} \left(\frac{1}{1-\varepsilon}\right)^{n-|S|} \left(1-\frac{\varepsilon}{1-\varepsilon}\right)^{|S|} \\
&= \frac{(1-2\varepsilon)^n}{N}
\end{aligned}
$$

# Fourier Coefficients of the Noise Operator: Second Technique I

- We shall prove a much more general result using a significantly (conceptually) simpler proof
- For $1 \leqslant i \leqslant n$, define the noise operator $N_{\varepsilon,i}$ that flips the $i$-th bit of $0^n$ with probability $\varepsilon$; otherwise keeps it intact. So, we have

$$N_{\varepsilon,i}(x) = \begin{cases} (1-\varepsilon), & x = 0 \\ \varepsilon, & x = \delta_i \\ 0, & \text{otherwise} \end{cases}$$

- Therefore, we have

$$\widehat{N_{\varepsilon,i}}(S) = \frac{1}{N}\left((1-\varepsilon) + \varepsilon(-1)^{S\cdot\delta_i}\right) = \frac{(1-2\varepsilon)^{S_i}}{N}$$

# Fourier Coefficients of the Noise Operator: Second Technique II

- Now, consider the noise operator $N_{\varepsilon,i} \oplus N_{\varepsilon,j}$. This noise operator only flips the $i$-th and the $j$-th bits of $0^n$ independently with probability $\varepsilon$. Since, we know that $A \oplus B = N(A * B)$ and $\widehat{(A * B)}(S) = \widehat{A}(S)\widehat{B}(S)$, we have the following result

$$\left(N_{\varepsilon,i} \widehat{\oplus} N_{\varepsilon,j}\right)(S) = \frac{(1-2\varepsilon)^{S_i+S_j}}{N}$$

- Applying this result inductively, we obtain

$$\left(N_{\varepsilon,1} \widehat{\oplus \cdots \oplus} N_{\varepsilon,n}\right)(S) = \frac{(1-2\varepsilon)^{S_1+\cdots+S_n}}{N} = \frac{(1-2\varepsilon)^{|S|}}{N}$$

- Note that $N_\varepsilon$ is identical to the distribution $N_{\varepsilon,1} \oplus \cdots \oplus N_{\varepsilon,n}$

# Noisy Version of a Function

- Let $f\colon \{0,1\}^n \to \mathbb{R}$ be a function
- Define the noisy-version of $f$ (represented as $\widetilde{f} = T_\rho(f)$) as follows

$$\widetilde{f}(x) = T_\rho(f) := \mathbb{E}\left[f(x+e)\colon e \sim N_\varepsilon\right],$$

  where $\rho = 1 - 2\varepsilon$
- So, we have

$$\widetilde{f}(x) = \sum_{e \in \{0,1\}^n} N_\varepsilon(e) f(x+e)$$

- So, over the domain $\{0,1\}^n$, this observation implies

$$\widetilde{f} = N(N_\varepsilon * f)$$

- So, we have $\widehat{\left(\widetilde{f}\right)}(S) = (1-2\varepsilon)^{|S|}\widehat{f}(S) = \rho^{|S|}\widehat{f}(S)$. We conclude that $\widehat{T_\rho(f)}(S) = \rho^{|S|}\widehat{f}(S)$.
- Intuitively, a Fourier coefficient of $\widetilde{f}$ is a damped version of the respective Fourier coefficient of $f$. Moreover, the dampening is proportional to the weight of $S$